

# Inhomogeneous Boltzmann equations: distance, asymptotics and comparison of the classical and quantum cases

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## Abstract

The notion of distance between a global Maxwellian function and an arbitrary solution  $f$  (with the same total density  $\rho$  at the fixed moment  $t$ ) of Boltzmann equation is introduced. In this way we essentially generalize the important Kullback-Leibler distance, which was used before. Namely, we generalize it for the spatially inhomogeneous case. An extremal problem to find a solution of the Boltzmann equation, such that  $\text{dist}\{M, f\}$  is minimal in the class of solutions with the fixed values of energy and of  $n$  moments, is solved. The cases of the classical and quantum (for Fermi and Bose particles) Boltzmann equations are studied and compared. The asymptotics and stability of solutions of the Boltzmann equations are also considered.

## 1 Introduction.

We consider the classical and quantum versions of Boltzmann equations (where the quantum version contains both the fermion and boson cases). The important notion of Kullback-Leibler distance [6], which was fruitfully

used before (see further references in the recent papers [4, 17, 20]), is essentially generalized and new conventional extremal problems, which appear in this way, are solved. The solution  $f(t, x, \zeta)$  of the Boltzmann equation is studied in the bounded domain  $\Omega$  of the  $x$ -space. Such an approach essentially changes the usual situation, that is, the total energy depends on  $t$  and the notion of distance between a stationary solution and an arbitrary solution of the Boltzmann equation includes the  $x$ -space. Thus, the notion of distance remains well-defined in the spatially inhomogeneous case too. Recall that the Kullback-Leibler distance is defined only in the spatially homogeneous case. The comparison of the classical and quantum mechanics, which was treated in [12–14], is generalized here for the case of the Boltzmann equations. It is especially interesting for the applications that the fermion and boson cases are essentially different from this point of view. In the last section of the paper we introduce the dissipative and conservative solutions and find the conditions under which the stationary solution of the classical Boltzmann equation is stable.

First, we discuss the classical case. The well-known classical Boltzmann equation for the monoatomic gas has the form

$$\frac{\partial f}{\partial t} = -\zeta \cdot \nabla_x f + Q(f, f), \quad (1.1)$$

where  $t \in \mathbb{R}$  stands for time,  $x = (x_1, \dots, x_n) \in \Omega$  stands for space coordinates,  $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$  is velocity, and  $\mathbb{R}$  denotes the real axis. The collision operator  $Q$  is defined by the relation

$$Q(f, f) = \int_{\mathbb{R}^n} \int_{S^{n-1}} B(\zeta - \zeta_*, \sigma) [f(\zeta') f(\zeta'_*) - f(\zeta) f(\zeta_*)] d\sigma d\zeta_*, \quad (1.2)$$

where  $B(\zeta - \zeta_*, \sigma) \geq 0$  is the collision kernel. Here we used the notation

$$\zeta' = (\zeta_* + \zeta)/2 + \sigma|\zeta_* - \zeta|/2, \quad \zeta'_* = (\zeta_* + \zeta)/2 - \sigma|\zeta_* - \zeta|/2, \quad (1.3)$$

where  $\sigma \in S^{n-1}$ , that is,  $\sigma \in \mathbb{R}^n$  and  $|\sigma| = 1$ . The solution  $f(t, x, \zeta)$  of Boltzmann equation (1.1) is the distribution function of gas. We start with some global Maxwellian function  $M$ , which is the stationary solution (with the total density  $\rho$ ) of the Boltzmann equation. The notion of distance between the global Maxwellian function and an arbitrary solution  $f$  (with the same value  $\rho$  of the total density at the fixed moment  $t$ ) of the Boltzmann equation is introduced. As already mentioned before, our approach enables us to

treat also the inhomogeneous case. An extremal problem to find a solution of the Boltzmann equation, such that  $\text{dist}\{M, f\}$  is minimal in the class of solutions with the fixed values of energy and of  $n$  moments, is solved.

The same considerations prove fruitful for the quantum Boltzmann equation. Our definition of the quantum entropy  $S_q$  is slightly different from the previous definitions (see [2, 10]). We show that the natural requirement

$$S_q \rightarrow S_c, \quad \varepsilon \rightarrow 0 \quad (S_c \text{ is the classical entropy}) \quad (1.4)$$

is not fulfilled in the case of old definition, however (1.4) holds in the case of our modified definition (see Section 6).

Some necessary preliminary definitions and results are given in Section 2. An important functional, which attains maximum at the global Maxwellian function is introduced in Section 3. The distance between solutions of (1.1) and the corresponding extremal problem are studied in Section 4. The modified Boltzmann equations for Fermi and Bose particles (the quantum cases) are considered in Sections 5 and 6. A comparison of the classical and quantum cases is also conducted in Section 6. Finally, Section 7 is dedicated to the asymptotics and stability of solutions.

We use the notation  $C_0^1$  to denote the class of differentiable functions  $f(\zeta)$ , which tend to zero sufficiently rapidly when  $\zeta$  tends to infinity.

## 2 Preliminaries: basic definitions and results

In this section we present some well-known notions and results connected with the Boltzmann equation. The distribution function  $f(t, x, \zeta)$  is non-negative:

$$f(t, x, \zeta) \geq 0, \quad (2.1)$$

and so the entropy

$$S(t, f) = - \int_{\Omega} \int_{\mathbb{R}^n} f(t, x, \zeta) \log f(t, x, \zeta) d\zeta dx \quad (2.2)$$

is well-defined.

**Definition 2.1** *A function  $\phi(\zeta)$  is called a collision invariant if it satisfies the relation*

$$\int_{\mathbb{R}^n} \phi(\zeta) Q(f, f)(\zeta) d\zeta = 0 \quad \text{for all } f \in C_0^1. \quad (2.3)$$

It is well-known (see [19]) that there are the following collision invariants:

$$\phi_0(\zeta) = 1, \quad \phi_i(\zeta) = \zeta_i \quad (i = 1, 2, \dots, n), \quad \phi_{n+1}(\zeta) = |\zeta|^2. \quad (2.4)$$

The notions of density  $\rho(t, x)$ , total density  $\rho(t)$ , mean velocity  $u(t, x)$ , energy  $E(t, x)$ , and total energy  $E(t)$  are introduced via formulas:

$$\rho(t, x) = \int_{\mathbb{R}^n} f(t, x, \zeta) d\zeta, \quad \rho(t) = \int_{\Omega} \rho(t, x) dx, \quad (2.5)$$

$$u(t, x) = (1/\rho(t, x)) \int_{\mathbb{R}^n} \zeta f(t, x, \zeta) d\zeta, \quad (2.6)$$

$$E(t, x) = \int_{\mathbb{R}^n} \frac{|\zeta|^2}{2} f(t, x, \zeta) d\zeta, \quad E(t) = \int_{\Omega} \int_{\mathbb{R}^n} \frac{|\zeta|^2}{2} f(t, x, \zeta) d\zeta dx. \quad (2.7)$$

The function

$$f = (\rho/(2\pi T)^{n/2}) \exp(-|\zeta - u|^2/(2T)). \quad (2.8)$$

is called the *global Maxwellian* and is a function of the mass density  $\rho > 0$ , bulk velocity  $u = (u_1, \dots, u_n)$  and temperature  $T$ . We assume that the domain  $\Omega$  is bounded and so its volume is bounded too:

$$\text{Vol}(\Omega) = V_{\Omega} < \infty. \quad (2.9)$$

Therefore, the function

$$M(\zeta) = (\rho/(V_{\Omega}(2\pi T)^{n/2})) \exp(-|\zeta - u|^2/(2T)) \quad (2.10)$$

is a global Maxwellian with the constant total density  $\rho$ .

**Proposition 2.1** [19] *The global Maxwellian function  $M(\zeta)$  is the stationary solution of the Boltzmann equation (1.1).*

Boltzmann proved in [1] the fundamental result below:

**Theorem 2.1** *Let  $f \in C_0^1$  be a non-negative solution of equation (1.1). Then the following inequality holds:*

$$dS/dt \geq 0. \quad (2.11)$$

### 3 Extremal problem

Similar to the cases considered in [14, 15], an important role is played by the functional

$$F(f) = (F(f))(t) = \lambda E(t) + S(t), \quad \lambda = -1/T, \quad (3.1)$$

where  $S$  and  $E$ , respectively, are defined by formulas (2.2) and (2.7), and the functional (3.1) is considered on the class of functions with the same  $\rho(t) = \rho$  at the fixed moment  $t$ . The parameters  $\lambda = -1/T$  and  $\rho$  are fixed.

Now, we use the calculus of variations (see [5]) and find the function  $f_{\max}$  which maximizes the functional (3.1). The corresponding Euler's equation takes the form

$$\frac{\delta}{\delta f} \left[ \lambda \frac{|\zeta|^2}{2} f - f \log f + \mu f \right] = 0. \quad (3.2)$$

Here  $\frac{\delta}{\delta f}$  stands for the functional derivative. Our extremal problem is conditional and  $\mu$  is the Lagrange multiplier. Hence, we have

$$\lambda \frac{|\zeta|^2}{2} - 1 - \log f + \mu = 0. \quad (3.3)$$

From the last relation we obtain

$$f = C e^{-|\zeta|^2/(2T)}. \quad (3.4)$$

Formulas (2.10) and (3.4) imply that

$$f = M(\zeta) = \frac{\rho}{V_{\Omega}(2\pi T)^{n/2}} e^{-\frac{|\zeta|^2}{2T}}. \quad (3.5)$$

In view of (2.2), (2.7), and (3.1) we see that

$$F(f) = \int_{\Omega} \int_{\mathbb{R}^n} L_f(t, x, \zeta) d\zeta dx, \quad L_f = -\left( \frac{|\zeta|^2}{2T} + \log f \right) f. \quad (3.6)$$

For positive  $f$  (including the case  $f = M$ ) and for  $L_f$  given in (3.6), we have the inequality

$$\frac{\delta^2}{\delta f^2} L_f = -1/f < 0. \quad (3.7)$$

**Corollary 3.1** *The global Maxwellian function  $M(\zeta)$ , which is defined by formula (3.4), gives the maximum of the functional  $F$  on the class of functions with the same value  $\rho$  of the total density  $\rho(t)$  at the fixed moment  $t$ .*

It follows from (2.5), (3.5), and (3.6) that

$$F(M) = -\rho \log \left( \frac{\rho}{V_\Omega (2\pi T)^{n/2}} \right). \quad (3.8)$$

Therefore, Corollary 3.1 can also be proved without using the calculus of variation (see [18]). Indeed, taking into account relations (3.5), (3.6), and (3.8) and the fact that the total densities of  $M$  and  $f$  are equal, we have

$$F(M) - F(f) = \int_\Omega \int_{\mathbb{R}^n} M \left( 1 - \frac{f}{M} + \frac{f}{M} \log \frac{f}{M} \right) d\zeta dx. \quad (3.9)$$

Using inequality  $1 - x + x \log x > 0$  for  $x > 0$ ,  $x \neq 1$ , we derive from (3.9) that

$$F(M) - F(f) > 0 \quad (f \neq M). \quad (3.10)$$

**Remark 3.1** *Since the extremal problem is conditional, the connection between the energy and entropy can be interpreted in terms of game theory. The functional (3.1) defines this game. The global Maxwellian function  $M(\zeta)$  is the solution of it. A game interpretation of quantum and classical mechanics problems is given in the papers [14, 15].*

**Remark 3.2** *Inequality (3.8) is valid for all the non-negative functions  $f$  with the fixed density  $\rho$  at  $t$  (not necessarily solutions of the Boltzmann equation).*

## 4 Distance

Let  $f(t, x, \zeta)$  be a nonnegative solution of the Boltzmann equation (1.1). We assume that  $T$  and the value  $\rho = \rho(t)$  at some moment  $t$  are fixed. According to (3.10) we have

$$F(M) - F(f) \geq 0, \quad (4.1)$$

where the global Maxwellian function  $M(\zeta)$  is defined in (3.5). The equality in (4.1) holds if and only if  $f(t, x, \zeta) = M(\zeta)$ . Hence, we can introduce the following definition of distance between the solution  $f(t, x, \zeta)$  and the global Maxwellian function  $M(\zeta)$ :

$$\text{dist}\{M, f\} = F(M) - F(f). \quad (4.2)$$

**Remark 4.1** *In the spatially homogeneous case (if not only the total densities  $\rho_M$  and  $\rho_f$  of  $M$  and  $f$  are equal but the energies  $E_M$  and  $E_f$  are equal too), our definition (4.2) of distance coincides with the Kullback-Leibler distance (see [20]). However, our approach enables us to treat also the inhomogeneous case.*

Next, we study the case  $E_M \neq E_f$  and start with an example.

**Example 4.1** *Let  $T_1 \neq T$  and consider the global Maxwellian function*

$$M_1(\zeta) = \frac{\rho}{V_\Omega(2\pi T_1)^{n/2}} \exp\left(-\frac{|\zeta|^2}{2T_1}\right). \quad (4.3)$$

*Direct calculation shows that*

$$E_1 = E_{M_1} = \rho n T_1 / 2 \neq E, \quad (4.4)$$

$$F(M_1) = -\rho \left( \log\left(\frac{\rho}{V_\Omega(2\pi T_1)^{n/2}}\right) - n(1 - T_1/T)/2 \right). \quad (4.5)$$

*It follows from (3.8) and (4.5) that*

$$\text{dist}\{M, M_1\} = -\rho n (\log(T_1/T) - T_1/T + 1)/2. \quad (4.6)$$

We introduce the class  $C(\rho, E_1, U)$  of non-negative functions  $f(t, x, \zeta)$  with the given total density  $\rho$  (see (2.8)), total energy

$$\int_\Omega \int_{\mathbb{R}^n} \frac{|\zeta|^2}{2} f(t, x, \zeta) d\zeta dx = E_1, \quad (4.7)$$

and total moments  $U = (U_1, U_2, \dots, U_n)$ , where

$$U_k = \int_\Omega \int_{\mathbb{R}^n} \zeta_k f(t, x, \zeta) d\zeta dx. \quad (4.8)$$

Recall that the global Maxwellian function  $M$  is defined by (3.5).

**Extremal problem.** *Find a function  $f$ , which minimizes the functional  $\text{dist}\{M, f\}$  on the class  $C(\rho, E_1, U)$ .*

The corresponding Euler's equation takes the form

$$\frac{\delta}{\delta f} \left[ (\lambda + \nu) \frac{|\zeta|^2}{2} f - f \log f + \mu f + f \sum_k \gamma_k \zeta_k \right] = 0. \quad (4.9)$$

Recall that our extremal problem is conditional, and  $\mu, \nu, \gamma_k$  are the Lagrange multipliers. Hence, we have

$$(\lambda + \nu) \frac{|\zeta|^2}{2} - \log f - 1 + \mu + \sum_k \gamma_k \zeta_k = 0. \quad (4.10)$$

From the last relation we obtain

$$f = C \exp \left( (\lambda + \nu) \frac{|\zeta|^2}{2} + \sum_k \gamma_k \zeta_k \right). \quad (4.11)$$

According to (2.5) we have  $\lambda + \nu < 0$ . Now, we rewrite (4.11) as

$$f = C_1 \left( -\frac{2\pi}{\lambda + \nu} \right)^{-n/2} \exp \left( \frac{\lambda + \nu}{2} \sum_k \left( \zeta_k + \frac{\gamma_k}{\lambda + \nu} \right)^2 \right), \quad (4.12)$$

where

$$C_1 = C \frac{\pi^{n/2}}{(-(\lambda + \nu)/2)^{n/2}} \exp \left( -\frac{\sum_k \gamma_k^2}{2(\lambda + \nu)} \right). \quad (4.13)$$

To calculate the parameters  $\mu, \nu, \gamma_k$  we use again the well-known formulas

$$\int_{-\infty}^{\infty} e^{-a\xi^2} d\xi = \sqrt{\pi/a}, \quad \int_{-\infty}^{\infty} \xi^2 e^{-a\xi^2} d\xi = \frac{1}{2a} \sqrt{\pi/a}, \quad a > 0. \quad (4.14)$$

Formulas (2.5), (4.7), (4.8), (4.12), and (4.14) imply that

$$C_1 = \rho/V_\Omega, \quad \gamma_k/(\lambda + \nu) = -U_k/\rho, \quad -(\lambda + \nu) = T_1^{-1}, \quad (4.15)$$

where

$$T_1 = \frac{2}{n\rho} E_1 - \frac{1}{n\rho^2} \sum_k U_k^2. \quad (4.16)$$



Because of (4.12) and (4.15) we see that  $f$  is just another global Maxwellian function

$$f = M_2(\zeta) = \frac{\rho}{V_\Omega(2\pi T_1)^{n/2}} \exp\left(-\frac{|\zeta - U/\rho|^2}{2T_1}\right). \quad (4.17)$$

In the same way as (4.5) we obtain:

$$F(M_2) = -\rho \left( \log\left(\frac{\rho}{V_\Omega(2\pi T_1)^{n/2}}\right) - n(1 - T_1/T)/2 \right) - \frac{1}{2\rho T} |U|^2. \quad (4.18)$$

Moreover, formulas (3.6) and (4.2) imply the relations

$$\text{dist}\{M, f\} = \int_\Omega \int_{\mathbb{R}^n} (L_M(t, x, \zeta) - L_f(t, x, \zeta)) d\zeta dx, \quad \frac{\delta^2}{\delta f^2} (L_M - L_f) = 1/f. \quad (4.19)$$

That is, the functional  $\text{dist}\{M, f\}$  attains its minimum on the function  $f = M_2$ , which satisfies conditions  $\rho(t) = \rho$ , (4.7), and (4.8). More precisely, in view of (4.18) we have

$$\text{dist}\{M, M_2\} = -\frac{n\rho}{2} (\log(T_1/T) - T_1/T + 1) + \frac{|U|^2}{2\rho T}. \quad (4.20)$$

Hence, the following assertion is valid.

**Proposition 4.1** *Let  $M$  and  $M_2$ , respectively, be defined by (3.5) and (4.17). If the function  $f$  satisfies conditions  $\rho(t) = \rho$ , (4.7), (4.8), and  $f \neq M_2$ , then*

$$\text{dist}\{M, f\} > -\frac{n\rho}{2} (\log(T_1/T) - T_1/T + 1) + \frac{|U|^2}{2\rho T}.$$

**Definition 4.1** *We denote by  $\widehat{M}$  the Maxwell function of the form (3.5), where  $\rho = (1/e)(2\pi T)^{n/2} V_\Omega$ .*

According to (3.8) we have

$$F(\widehat{M}) > F(M), \quad M \neq \widehat{M}. \quad (4.21)$$

Hence the following statement is valid.

**Proposition 4.2** *The inequality*

$$\widehat{G}(f) = F(\widehat{M}) - F(f) > 0, \quad f \neq \widehat{M} \quad (4.22)$$

*is fulfilled for all non-negative  $f$ .*

We call  $\widehat{G}$  in (4.22) the *Lyapunov functional*, and will study it in greater detail in Section 7.

## 5 Modified Boltzmann equations for Fermi and Bose particles.

We study the modified Boltzmann equation which takes into account the quantum effect [2, 9]:

$$\frac{\partial f}{\partial t} = -\zeta \cdot \nabla_x f + C(f, f). \quad (5.1)$$

The collision operator  $C$  is defined by the relation

$$C(f, f) = \int_{\mathbb{R}^n} \int_{S^{n-1}} B(\zeta - \zeta_*, \sigma) [f(\zeta') f(\zeta'_*) (1 + \varepsilon f(\zeta)) (1 + \varepsilon f(\zeta_*)) - f(\zeta) f(\zeta_*) (1 + \varepsilon f(\zeta')) (1 + \varepsilon f(\zeta'_*))] d\sigma d\zeta_*, \quad (5.2)$$

where  $\zeta'$  and  $\zeta'_*$  are introduced in (1.3), and  $\varepsilon \in \mathbb{R}$ . If  $\varepsilon = 0$ , the right-hand side of (5.2) coincides with the right-hand side of (1.2), that is, we get the classical case. The inequalities  $\varepsilon > 0$  and  $\varepsilon < 0$  hold for bosons and fermions, respectively. Similar to the classical case the quantum density  $\rho_\varepsilon$  and quantum energy  $E_\varepsilon$  are given by formulas (2.4) and (2.7), respectively. However, the quantum entropy  $S(t, \varepsilon)$  ( $\varepsilon \neq 0$ ) is defined in a more complicated way:

$$S(t, f, \varepsilon) = - \int_{\Omega} \int_{\mathbb{R}^n} [f \log f - (1/\varepsilon)(1 + \varepsilon f) \log(1 + \varepsilon f) + f] d\zeta dx \quad (5.3)$$

**Remark 5.1** *Our definition (5.3) of entropy is slightly different from the previous definitions (see [2, 10]). Namely, formula (5.3) contains the additional summand*

$$- \rho_\varepsilon = - \int_{\Omega} \int_{\mathbb{R}^n} f d\zeta dx. \quad (5.4)$$

*We shall show that the natural requirement*

$$S(\varepsilon) \rightarrow S_c, \quad \varepsilon \rightarrow 0 \quad (5.5)$$

*is fulfilled only in the case that (5.3) holds.*

## 6 Modified extremal problem

1. We assume again that the domain  $\Omega$  is bounded and introduce the functional

$$F_\varepsilon(f) = \lambda E_\varepsilon(f) + S(f, \varepsilon), \quad \lambda = -1/T, \quad (6.1)$$

where  $E_\varepsilon(f)$  and  $S(f, \varepsilon)$  are defined by formulas (2.7) and (5.3) respectively. The parameters  $\lambda = -1/T$  and  $\rho$  are fixed.

Again we use the calculus of variations (see [2]) and find the function  $f_{max}$  which maximizes the functional (6.1) under additional condition

$$\int_{\Omega} \rho(t, x) dx = \rho. \quad (6.2)$$

The corresponding Euler's equation takes the form

$$\lambda \frac{|\zeta|^2}{2} - \log f + \log(1 + \varepsilon f) - 1 + \mu = 0. \quad (6.3)$$

From the last relation we obtain

$$f/(1 + \varepsilon f) = C e^{-\frac{|\zeta|^2}{2T}}. \quad (6.4)$$

Formula (6.4) implies that

$$f = M_\varepsilon = \frac{C e^{-\frac{|\zeta|^2}{2T}}}{1 - C \varepsilon e^{-\frac{|\zeta|^2}{2T}}}. \quad (6.5)$$

It is required that the distribution  $M_\varepsilon$  is positive, that is,

$$C > 0, \quad -\infty < C \varepsilon \leq 1, \quad (6.6)$$

and further we assume that (6.6) holds. Moreover, (6.6) yields also the positivity of  $1 + \varepsilon M_\varepsilon$ :

$$M_\varepsilon(\zeta) > 0, \quad 1 + \varepsilon M_\varepsilon(\zeta) > 0. \quad (6.7)$$

According to (2.5) and (6.2), the constant  $C$  is defined by the equality

$$V_\Omega \int_{\mathbb{R}^n} \frac{C e^{-\frac{|\zeta|^2}{2T}}}{1 - \varepsilon C e^{-\frac{|\zeta|^2}{2T}}} d\zeta = \rho. \quad (6.8)$$

In view of (6.1), we have the relation which is similar to (3.6):

$$F_\varepsilon(f) = \int_{\Omega} \int_{\mathbb{R}^n} L_{f,\varepsilon}(t, x, \zeta) d\zeta dx. \quad (6.9)$$

Though the function  $L_{f,\varepsilon}$  is more complicated than  $L_f$  in (3.6), we easily get an analog of (3.7):

$$\frac{\delta^2}{\delta f^2} L_{f,\varepsilon} = -\frac{1}{f(1+\varepsilon f)} < 0, \quad (6.10)$$

which clearly holds if  $f$  and  $1+\varepsilon f$  are positive, including the case that  $f = M_\varepsilon$ .

**Corollary 6.1** *The functional  $F_\varepsilon$  given by (6.1) attains its maximum (for positive functions  $f$  satisfying condition (6.8)) on the function  $M_\varepsilon$  of the form (6.5). That is, for the distance  $G_\varepsilon$  we get*

$$G_\varepsilon(f) := F_\varepsilon(M_\varepsilon) - F_\varepsilon(f) > 0 \quad (f \neq M_\varepsilon). \quad (6.11)$$

**Remark 6.1** *The global Maxwellians  $M_\varepsilon$  play an essential role in boson and fermion theories. When the standard approach is used, they appear in a more complicated way (see [7, Ch.V, sections 52, 53] and [3, Ch.1, sections 9, 10]).*

**2.** Using the spherical coordinates, we calculate the integral on the left-hand side of (6.8)

$$\int_{\mathbb{R}^n} \frac{C e^{-\frac{|\zeta|^2}{2T}}}{1 - \varepsilon C e^{-\frac{|\zeta|^2}{2T}}} d\zeta = \omega_{n-1} C \int_0^\infty \frac{r^{n-1} e^{-\frac{r^2}{2T}}}{1 - \varepsilon C e^{-\frac{r^2}{2T}}} dr, \quad \omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad (6.12)$$

where  $\omega_{n-1}$  is the surface area of the  $(n-1)$ -sphere of radius 1, and  $\Gamma(z)$  is the Euler's Gamma function. Taking into account (6.8) and (6.12) we obtain

$$(2\pi T)^{n/2} V_\Omega C L_{n/2}(C\varepsilon) = \rho, \quad (6.13)$$

where

$$L_{n/2}(z) = \frac{2}{(2T)^{n/2} \Gamma(n/2)} \int_0^\infty \frac{r^{n-1} e^{-\frac{r^2}{2T}}}{1 - z e^{-\frac{r^2}{2T}}} dr. \quad (6.14)$$

Because of the equality

$$\int_0^\infty e^{-ar^2} r^{n-1} dr = \frac{1}{2} a^{-n/2} \Gamma(n/2) \quad (6.15)$$

the function  $L_{n/2}(z)$  admits the expansion

$$L_{n/2}(z) = \sum_{m=1}^{\infty} (z^{m-1} / m^{n/2}), \quad (6.16)$$

which yields the next statement.

**Proposition 6.1** *The function  $L_{n/2}(z)$  monotonically increases for  $0 < z < 1$  and*

$$L_{1/2}(1) = L_1(1) = \infty, \quad L_{n/2}(1) < \infty \quad \text{for } n > 2. \quad (6.17)$$

**Remark 6.2** *It is easy to see that  $L_{n/2}(z) = \infty$  for  $z > 1$ .*

In view of Proposition 6.1 we have:

**Corollary 6.2** *If  $\varepsilon > 0$  (boson case) and either  $n = 1$  or  $n = 2$ , then equation (6.13) has one and only one solution  $C$  such that  $C > 0$ ,  $C\varepsilon < 1$ , and so (6.6) holds.*

**Corollary 6.3** *If  $\varepsilon > 0$  (boson case),  $n > 2$  and*

$$(2\pi T)^{n/2} V_{\Omega} L_{n/2}(1) > \varepsilon \rho, \quad (6.18)$$

*then equation (6.13) has one and only one solution  $C$  such that  $C > 0$  and  $C\varepsilon < 1$ . If, instead of (6.18), we have  $(2\pi T)^{n/2} V_{\Omega} L_{n/2}(1) = \varepsilon \rho$ , then the solution of (6.13) is given by  $C = 1/\varepsilon$  and the corresponding  $M_{\varepsilon}$  has singularity at  $\zeta = 0$ .*

**Remark 6.3** *The function  $L_{n/2}(z)$  belongs to the class of the  $L$ -functions [8] and is connected with the famous Riemann zeta-function*

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}; \quad \Re z > 1 \quad (6.19)$$

*by the relation*

$$L_{n/2}(1) = \zeta(n/2). \quad (6.20)$$

*Hence, some useful estimates for  $L_{n/2}(1)$  follow. In particular, we get*

$$L_{3/2}(1) = 2.612, \quad L_2(1) = 1.645, \quad L_{5/2}(1) = 1.341, \quad L_3(1) = 1.202. \quad (6.21)$$

Let us consider the fermion case (i.e., the case  $\varepsilon < 0$ ). The next proposition easily follows from (6.14) and monotonical increase of  $ax(1+ax)^{-1}$  ( $a > 0$ ) on the positive half-axis.

**Proposition 6.2** *Let  $\varepsilon < 0$ . Then the function  $CL_{n/2}(C\varepsilon)$  monotonically increases with respect to  $C > 0$ . Furthermore, we have  $CL_{n/2}(C\varepsilon) \rightarrow \infty$  for  $C \rightarrow \infty$ .*

**Corollary 6.4** *If  $\varepsilon < 0$  (fermion case), then equation (6.13) has one and only one solution  $C$  such that  $C > 0$ .*

3. Consider now the energy for the global Maxwellian  $M_\varepsilon$ :

$$E_\varepsilon(M_\varepsilon) = \int_{\Omega} \int_{\mathbb{R}^n} \frac{|\zeta|^2 C e^{-\frac{|\zeta|^2}{2T}}}{1 - \varepsilon C e^{-\frac{|\zeta|^2}{2T}}} d\zeta dx / 2 = V_{\Omega} \omega_{n-1} C \int_0^\infty \frac{r^{n+1} e^{-\frac{r^2}{2T}}}{1 - \varepsilon C e^{-\frac{r^2}{2T}}} dr / 2. \quad (6.22)$$

Formulas (6.12)–(6.14) and (6.22) imply that

$$E_\varepsilon(M_\varepsilon) = \left( \frac{n\rho T}{2} \right) \frac{L_{n/2+1}(C\varepsilon)}{L_{n/2}(C\varepsilon)}. \quad (6.23)$$

According to (4.4) the corresponding classical energy  $E = E_0 = E_c$  is given by the formula

$$E_c(M) = \frac{n\rho T}{2} \quad (M = M_0). \quad (6.24)$$

**Proposition 6.3** *If  $\varepsilon > 0$  (boson case), then we have*

$$E_\varepsilon < E_c. \quad (6.25)$$

*If  $\varepsilon < 0$  (fermion case) and*

$$\text{either } n \geq 2, \quad -C\varepsilon \leq 1 \quad \text{or} \quad n = 1, \quad -C\varepsilon < 3^{3/2}/2^{5/2} \approx 0.91, \quad (6.26)$$

*then we have*

$$E_c < E_\varepsilon. \quad (6.27)$$

*Proof.* Taking into account (6.16), we obtain  $L_{n/2+1}(C\varepsilon)/L_{n/2}(C\varepsilon) < 1$  for  $\varepsilon > 0$ . Hence, in view of (6.23) and (6.24) the inequality (6.25) holds in the boson case.

If  $\varepsilon < 0$  and conditions (6.26) hold, the inequalities

$$L_{n/2}(C\varepsilon) > 0 \quad \text{and} \quad L_{n/2+1}(C\varepsilon) - L_{n/2}(C\varepsilon) > 0$$

follow from (6.14) and (6.16), respectively, and we get

$$L_{n/2+1}(C\varepsilon)/L_{n/2}(C\varepsilon) > 1.$$

That is, in view of (6.23) and (6.24) the inequality (6.27) is proved in the fermion case.  $\square$

4. For the classical case  $\varepsilon = 0$  formula (6.13) (see also (2.10)) implies

$$C = C_0 = \rho/V_\Omega (2\pi T)^{n/2}. \quad (6.28)$$

In view of (3.1), (3.8), and (6.28) we easily derive for  $M = M_0$  that

$$S_c = \frac{1}{T} E_c - \rho \log C_0. \quad (6.29)$$

To calculate the quantum entropy  $S(M_\varepsilon, \varepsilon)$  we recall (6.5) and use equalities

$$M_\varepsilon = g/(1 - \varepsilon g), \quad 1 + \varepsilon M_\varepsilon = (1 - \varepsilon g)^{-1}, \quad g := C e^{-|\zeta|^2/(2T)} \quad (6.30)$$

to simplify the expression, which stands under integral on the right-hand side of (5.3) and which we denote by  $L_S$ :

$$L_S = M_\varepsilon(1 + \log g) + (1/\varepsilon) \log(1 - \varepsilon g). \quad (6.31)$$

Substitute  $\log g = \log C - (1/2T)|\zeta|^2$  into (6.31) and substitute (6.31) into (5.3) to get

$$S(M_\varepsilon, \varepsilon) = \frac{1}{T} E_\varepsilon - (1 + \log C)\rho - \frac{1}{\varepsilon} V_\Omega \int_{\mathbb{R}^n} \log(1 - \varepsilon g) d\zeta. \quad (6.32)$$

Using integration by parts and the definition (2.7) of energy we rewrite (6.32):

$$S(M_\varepsilon, \varepsilon) = \frac{1}{T} E_\varepsilon - (1 + \log C)\rho + \frac{2E_\varepsilon}{nT}. \quad (6.33)$$

From (6.1), (6.24), (6.29), and (6.33) we see that

$$S(M_\varepsilon, \varepsilon) - S_c = \frac{n+2}{nT} (E_\varepsilon - E_c) - \rho \log(C/C_0), \quad (6.34)$$

$$F_\varepsilon - F_c = \frac{2}{nT} (E_\varepsilon - E_c) - \rho \log(C/C_0) \quad (F_c = F_0). \quad (6.35)$$

The behavior of  $C$  is of interest and we start with the proposition below.

**Proposition 6.4** *The following inequalities are valid:*

$$C > C_0 \quad \text{for } \varepsilon < 0; \quad C < C_0 \quad \text{for } \varepsilon > 0. \quad (6.36)$$

Proof. According to (6.14) and (6.16) we have

$$L_{n/2}(z_1) < L_{n/2}(0) = 1 < L_{n/2}(z_2) \quad \text{for } z_1 < 0 < z_2 \leq 1. \quad (6.37)$$

Therefore, it is immediate that

$$C_0 L_{n/2}(C_0 \varepsilon_1) < C_0, \quad C_0 < C_0 L_{n/2}(C_0 \varepsilon_2) \quad \text{for } \varepsilon_1 < 0 < \varepsilon_2. \quad (6.38)$$

In view of Propositions 6.1 and 6.2 the functions  $CL_{n/2}(C\varepsilon_1)$  and  $CL_{n/2}(C\varepsilon_2)$  increase with respect to  $C > 0$ , and so formulas (6.13) and (6.38) imply (6.36).  $\square$

It is immediate from (6.36) that  $C$  is bounded for  $\varepsilon > 0$ . However,  $C$  is bounded also for the small values of  $|\varepsilon|$ , when  $\varepsilon$  is negative. Indeed, let  $-(2C_0)^{-1} < \varepsilon < 0$ . Then, formula (6.14) yields

$$2L_{n/2}(2C_0\varepsilon) > 2L_{n/2}(-1) > L_{n/2}(0) = 1.$$

Therefore, we have  $2C_0 L_{n/2}(2C_0\varepsilon) > C_0$ , which in view of Proposition 6.2 implies  $C < 2C_0$ .

Now, rewrite (6.13) as  $z = C_0\varepsilon/L_{n/2}(z)$ , where  $z = C\varepsilon$ , and note that  $\left| \frac{d}{dz} \frac{C_0\varepsilon}{L_{n/2}(z)} \right| < 1$  for  $|z| < 1$  and small values of  $\varepsilon$ . (Since  $C$  is bounded, we see that  $|z| < 1/2$  for the sufficiently small values of  $\varepsilon$ .) Thus, we apply iteration method to the equation  $z = C_0\varepsilon/L_{n/2}(z)$  and derive

$$C = C_0 + O(\varepsilon), \quad \varepsilon \rightarrow 0. \quad (6.39)$$

Next we note that formula (6.13) yields  $CL_{n/2}(C\varepsilon) = C_0$ . Therefore, taking into account (6.39) we get

$$C/C_0 = 1/L_{n/2}(C\varepsilon) = 1 - (C_0\varepsilon)/2^{n/2} + O(\varepsilon^2). \quad (6.40)$$

Moreover, from (6.40) we see that

$$\log(C/C_0) = -(C_0\varepsilon)/2^{n/2} + O(\varepsilon^2). \quad (6.41)$$

Using relations (6.16), (6.23), (6.24), and (6.39), we derive

$$E_\varepsilon - E_c = -\frac{n\rho TC_0\varepsilon}{4(2^{n/2})} + O(\varepsilon^2), \quad \varepsilon \rightarrow 0. \quad (6.42)$$

Because of (6.34), (6.35), (6.41), and (6.42), we get the next proposition.



**Proposition 6.5** *For  $\varepsilon \rightarrow 0$ , we have equality (6.42) as well as equalities below:*

$$S(M_\varepsilon, \varepsilon) - S_c = -\frac{(n-2)\rho C_0 \varepsilon}{4(2^{n/2})} + O(\varepsilon^2), \quad (6.43)$$

$$F_\varepsilon - F_c = \frac{\rho C_0 \varepsilon}{2(2^{n/2})} + O(\varepsilon^2). \quad (6.44)$$

**Corollary 6.5** *Let  $\varepsilon_1 < 0 < \varepsilon_2$  be small. Then*

$$S(M_{\varepsilon_2}, \varepsilon_2) < S_c < S(M_{\varepsilon_1}, \varepsilon_1) \quad \text{for } n > 2, \quad F_{\varepsilon_1} < F_c < F_{\varepsilon_2} \quad \text{for all } n. \quad (6.45)$$

**Remark 6.4** *We recall that in view of Proposition 6.3 the inequalities*

$$E_{\varepsilon_2} < E_c < E_{\varepsilon_1}, \quad \varepsilon_1 < 0 < \varepsilon_2 \quad (6.46)$$

*hold without the demand for  $\varepsilon_i$  to be small. Here  $E_{\varepsilon_2}$  corresponds to the boson and  $E_{\varepsilon_1}$  to the fermion case.*

**Remark 6.5** *We note that relations (6.44) as well as their physical interpretation are contained in the well-known book by L. Landau and E. Lifshitz [7, Section 55].*

**Conjecture 6.1** *Relation (6.27), which was proved for all  $-C\varepsilon \leq 1$  ( $\varepsilon < 0$ ) in the case that  $n \geq 2$ , is valid also for all  $-C\varepsilon \leq 1$  ( $\varepsilon < 0$ ) in the case that  $n = 1$ . We recall that (6.27) holds for  $n = 1$  and  $-C\varepsilon < 3^{3/2}/2^{5/2}$ . Moreover, (6.27) holds in the extremal case  $C\varepsilon = -1$ . Indeed, using (6.20), (6.21), the relation  $\zeta(1/2) \approx -1.46$  and the well-known equality (see, e.g., [8, p.17])*

$$L_s(-1) = \zeta(s)(1 - 2^{1-s}), \quad (6.47)$$

*we obtain*

$$L_{3/2}(-1) \approx 0.765, \quad L_{1/2}(-1) \approx 0.6. \quad (6.48)$$

*Hence,  $L_{3/2}(-1)/L_{1/2}(-1) > 1$  and the conjecture is proved for the case that  $C\varepsilon = -1$ .*

## 7 Lyapunov functional

### 7.1 Classical case

In this subsection we prolong the study of the classical Boltzmann equation (1.1) and assume that  $f(t, x, \zeta)$  is its non-negative solution. Using Gauss-Ostrogradsky formula we write

$$\int_{\Omega} \int_{\mathbb{R}^n} (|\zeta|^2/2) \zeta \cdot \nabla_x f d\zeta dx = \int_{\partial\Omega} \int_{\mathbb{R}^n} (|\zeta|^2/2) [\zeta \cdot n(x)] f d\zeta d\sigma = A(t, \Omega), \quad (7.1)$$

$$\int_{\Omega} \int_{\mathbb{R}^n} \zeta \cdot \nabla_x f d\zeta dx = \int_{\partial\Omega} \int_{\mathbb{R}^n} [\zeta \cdot n(x)] f d\zeta d\sigma = B(t, \Omega), \quad (7.2)$$

where  $\partial\Omega$  is the piecewise smooth boundary of  $\Omega$ , and the integral  $\int_{\partial\Omega} g d\sigma$  is the surface integral with  $n(x)$  being the outward unit normal to that surface,  $x \in \partial\Omega$ .

**Remark 7.1** Here  $A(t, \Omega)$  and  $B(t, \Omega)$  are the total energy flux and the total density flux through the surface  $\partial\Omega$  per unit time, respectively.

**Definition 7.1** We say that a non-negative solution  $f(t, x, \zeta)$  of (1.1) belongs to the class  $\mathcal{D}(\Omega)$  of dissipative functions, if  $A(t, \Omega) \geq 0$  for all  $t$ .

**Definition 7.2** We say that a non-negative solution  $f(t, x, \zeta)$  of (1.1) belongs to the class  $\mathcal{C}(\Omega)$  of conservative functions, if  $A(t, \Omega) = 0$  for all  $t$ .

Clearly we have  $\mathcal{C}(\Omega) \subset \mathcal{D}(\Omega)$ . We note that the same definitions are applicable in the quantum case.

**Proposition 7.1** If the inequality  $f(t, x, \zeta) \geq 0$  and condition  $f(t, x, \zeta) = f(t, x, -\zeta)$  for  $x \in \partial\Omega$  hold, then we have  $f(t, x, \zeta) \in \mathcal{C}(\Omega)$ .

*Proof.* Since  $\int_{\mathbb{R}^n} (|\zeta|^2/2) f(t, x, \zeta) \zeta d\zeta = 0$ , it follows that  $A(t, \Omega) \equiv 0$  for  $A$  which is given by (7.1).  $\square$

**Remark 7.2** The so called bounce-back condition  $f(t, x, \zeta) = f(t, x, -\zeta)$  means that particles arriving with a certain velocity to the boundary  $\partial\Omega$  will bounce back with the opposite velocity (see [19, p.16]).

**Corollary 7.1** The global Maxwell functions  $M$  of the form (3.5) belong to the conservative class  $\mathcal{C}(\Omega)$ .

Furthermore, the next assertion can be easily derived via direct calculation.

**Corollary 7.2** *The global Maxwell functions  $M$  of the form (2.10), also belong to  $\mathcal{C}(\Omega)$ .*

**Example 7.1** *The well-known and important Maxwellian diffusion example (see [19, p.16]) is described by the property*

$$f(t, x, \zeta) = \rho_-(x) M_b(\zeta) \quad \text{for } x \in \partial\Omega, \quad \zeta \cdot n(x) > 0, \quad (7.3)$$

where  $M_b(\zeta)$  has the form (3.5). When we have

$$\int_{\partial\Omega} \int_{\zeta \cdot n(x) > 0} (|\zeta|^2/2) [\zeta \cdot n(x)] f d\zeta d\sigma \geq \int_{\partial\Omega} \int_{\zeta \cdot n(x) < 0} (|\zeta|^2/2) |\zeta \cdot n(x)| f d\zeta d\sigma, \quad (7.4)$$

the function  $f$  in (7.3) is dissipative. If in relation (7.4) we have the equality, then  $f$  is conservative. Hence, such functions satisfy our statements below (and the results below are new even for this case).

Now, consider the Lyapunov functional  $\widehat{G}(f) = F(\widehat{M}) - F(f)$  for the equation (1.1). According to (4.22) we have

$$\widehat{G}(f) > 0 \quad \text{for } f \neq \widehat{M}, \quad \widehat{G}(\widehat{M}) = 0. \quad (7.5)$$

Using Theorem 2.1 we derive the following assertion.

**Theorem 7.1** *Let  $f \in C_0^1$  be a non-negative dissipative solution of (1.1). Then the inequality  $(d\widehat{G}/dt) \leq 0$  is valid.*

*Proof.* The function  $\phi(\zeta) = |\zeta|^2$  is a collision invariant (i.e., (2.3) holds). Therefore, taking into account (1.1), (7.1), and Definition 7.1 we have

$$\frac{d}{dt} \int_{\Omega} \int_{\mathbb{R}^n} (|\zeta|^2/2) f d\zeta dx = -A(t, \Omega) \leq 0, \quad (7.6)$$

that is,  $(dE/dt) \leq 0$ . Recall also that  $\widehat{M}$  is a stationary solution, and so  $d\widehat{G}/dt = -dF(f)/dt$ . Now, the assertion of the theorem follows from (2.11) and (3.1).  $\square$

According to Theorem 7.1, if its conditions are fulfilled and  $(\widehat{G}(f))(t_0) < \delta$ , then the inequality  $(\widehat{G}(f))(t) < \delta$  holds for all  $t > t_0$ . Thus, the following important result is proved.

**Theorem 7.2** *If the distance is defined by  $\widehat{G}$  and  $f$  is dissipative, the stationary solution  $\widehat{M}$  is locally stable.*

The previous results on local stability [9, 18] were obtained for the spatially homogeneous Boltzmann equation.

**Corollary 7.3** *Let conditions of Theorem 7.1 be fulfilled. Then the function  $F(f)$  monotonically increases with respect to  $t$  and is bounded. Hence, there is a limit*

$$\lim_{t \rightarrow \infty} F(f) = \Phi \leq F(\widehat{M}). \quad (7.7)$$

Next, assume that the following limits exist:

$$\rho_\infty = \lim_{t \rightarrow \infty} \rho(t) \neq 0, \quad U_\infty = \lim_{t \rightarrow \infty} U(t), \quad (7.8)$$

where  $\rho(t)$  and  $U(t)$  are given by (2.5) and (4.8), respectively. We see from (3.8) and (7.8) that the functions  $M$  and  $M(t)$  of the form (3.5), where  $\rho_\infty$  and  $\rho(t)$ , respectively, are substituted in place of  $\rho$ , satisfy relations

$$F(M) = -\rho_\infty \log \left( \frac{\rho_\infty}{V_\Omega (2\pi T)^{n/2}} \right) = \lim_{t \rightarrow \infty} F(M(t)). \quad (7.9)$$

**Proposition 7.2** *Let the relations (7.7) and (7.8) hold. Then we have the inequality*

$$F(M) - \Phi \geq |U_\infty|^2 / (2\rho_\infty T). \quad (7.10)$$

Moreover, if the inequality (7.10) turns into equality, there is a unique Maxwell function  $M_U$  of the form (4.17) (with  $\rho = \rho_\infty$  and  $U = U_\infty$ ) such that

$$F(M_U) = \Phi. \quad (7.11)$$

If the inequality (7.10) is strict, that is,

$$F(M) - \Phi > |U_\infty|^2 / (2\rho_\infty T), \quad (7.12)$$

there are two such functions ( $M_1$  and  $M_2$ ) satisfying

$$F(M_k) = \Phi \quad (k = 1, 2). \quad (7.13)$$

Proof. It is immediate that

$$y(x) := x - 1 - \log x = 0 \quad \text{for } x = 1, \quad y(x) > 0 \quad \text{for } x > 0, x \neq 1. \quad (7.14)$$

Since  $y \geq 0$ , according to Proposition 4.1 we have

$$F(M(t)) - F(f(t)) \geq |U(t)|/(2\rho(t)T). \quad (7.15)$$

In view of (7.7)-(7.9) and (7.15) we get (7.10).

Now, using (4.20) we rewrite equation (7.11) (or, correspondingly, (7.13)) in the form

$$\frac{2}{n\rho_\infty} \left( F(M) - \Phi - \frac{|U_\infty|^2}{2\rho_\infty T} \right) = x - 1 - \log x, \quad (7.16)$$

where  $M_U$  or, correspondingly,  $M_k$  are expressed via solutions  $x_k$  of (7.16) in the form (compare with (4.17)):

$$M_U = M_k = \frac{\rho_\infty}{V_\Omega(2\pi x_k T)^{n/2}} \exp \left( - \frac{|\zeta - U_\infty/\rho_\infty|^2}{2x_k T} \right).$$

According to (7.14), the equation (7.16) has a unique solution when (7.10) turns into equality and has two solutions when (7.10) is a strict inequality.

□

**Corollary 7.4** *Let the conditions of Proposition 7.2 be fulfilled. Then*

$$F(M_k) - F(f) \rightarrow 0, \quad t \rightarrow \infty. \quad (7.17)$$

**Corollary 7.5** *Let the conditions of Proposition 7.2 be fulfilled, where the strict inequality (7.12) holds. If the limit*

$$E_\infty = \lim_{t \rightarrow \infty} E(t) \quad (7.18)$$

*exists and the corresponding solution  $f(t, x, \zeta)$  converges to a Maxwell function, then either  $E_\infty = E_1$  or  $E_\infty = E_2$ .*

**Remark 7.3** *Proposition 7.2 and Corollaries 7.4 and 7.5 are valid if the limit (7.7) exists. We do not suppose there, that the corresponding solution  $f$  is dissipative.*

## 7.2 Quantum case

Now, we consider the quantum version (5.1) of the Boltzmann equation. The corresponding Lyapunov functional  $G_\varepsilon(f)$  has the form (6.11).

**Theorem 7.3** (see [2] and [18]) *Let  $f$  ( $f \in C_0^1$ ) be a non-negative solution of equation (6.1). Then the inequality*

$$\frac{dS_\varepsilon}{dt} \geq 0 \quad (7.19)$$

*is valid.*

In the same way as in the classical case we obtain the assertions.

**Theorem 7.4** *Let  $f \in C_0^1$  be a non-negative dissipative solution of equation (6.1). Then the inequality*

$$\frac{dG_\varepsilon}{dt} \leq 0 \quad (7.20)$$

*is valid.*

**Corollary 7.6** *Let the conditions of Theorem 7.4 be fulfilled. If*

$$G_\varepsilon(f(t_0, x, \xi)) < \delta, \quad (7.21)$$

*then*

$$G_\varepsilon(f(t, x, \xi)) < \delta, \quad t > t_0. \quad (7.22)$$

Thus, we proved that the stationary solution  $M_\varepsilon$  is *locally stable*, when the distance is defined by  $G_\varepsilon(f)$  and the function  $f$  is dissipative.

## 8 Conclusion

We see that the study of the Boltzmann equations in a bounded domain  $\Omega$  and the suggested new extremal problem allow us to introduce a notion of distance and obtain various results for the inhomogeneous classical and quantum cases. In particular, the notion of the dissipative solutions is introduced and asymptotics and stability of solutions of the classical and quantum Boltzmann equations is studied. Following, e.g., [11, 16] we plan also to consider solutions of the Boltzmann equations for the case of Tsallis entropy. The approach could be applied to other related equations, such as the Fokker-Planck equation.

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